

LONG TIME WELL-POSEDNESS OF PRANDTL SYSTEM WITH SMALL AND ANALYTIC INITIAL DATA

PING ZHANG AND ZHIFEI ZHANG

ABSTRACT. In this paper, we investigate the long time existence and uniqueness of small solution to d , for $d = 2, 3$, dimensional Prandtl system with small initial data which is analytic in the horizontal variables. In particular, we prove that d dimensional Prandtl system has a unique solution with the life-span of which is greater than $\varepsilon^{-\frac{4}{3}}$ if both the initial data and the value on the boundary of the tangential velocity of the outflow are of size ε . We mention that the tool developed in [4, 5] to make the analytical type estimates and the special structure of the nonlinear terms to this system play an essential role in the proof of this result.

Keywords: Prandtl system, Littlewood-Paley theory, life-span, energy method

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1. INTRODUCTION

In this paper, we investigate long time well-posedness to the following Prandtl system in $\mathbb{R}_+ \times \mathbb{R}_+^d$, for $d = 2, 3$, with small and analytic initial data:

$$(1.1) \quad \begin{cases} \partial_t u + u \cdot \nabla_h u + v \partial_y u - \partial_{yy} u + \nabla_h p = 0, & (t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^{d-1} \times \mathbb{R}_+, \\ \operatorname{div}_h u + \partial_y v = 0, \\ u|_{y=0} = 0, \quad v|_{y=0} = 0, \quad \text{and} \quad \lim_{y \rightarrow \infty} u(t, x, y) = U(t, x), \\ u|_{t=0} = u_0, \end{cases}$$

where $u = u$, $\nabla_h = \operatorname{div}_h = \partial_x$ for $d = 2$, and $u = (u^1, u^2)$, $\nabla_h = (\partial_{x_1}, \partial_{x_2})$, $\operatorname{div}_h = \partial_{x_1} + \partial_{x_2}$ for $d = 3$. (u, v) denotes the tangential and normal velocities of the boundary layer flow, $(U(t, x), p(t, x))$ are the values on the boundary of the tangential velocity and pressure of the outflow, which satisfies Bernoulli's law

$$\partial_t U + U \cdot \nabla_h U + \nabla_h p = 0 \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}^{d-1}.$$

This system proposed by Prandtl [20] is a model equation for the first order approximation of the velocity field near the boundary in the zero viscosity limit of the initial boundary problem of Navier-Stokes equations with the non-slip boundary condition. One may check [15, 7] and references therein for more introductions on boundary layer theory.

One of the key step to rigorously justify this inviscid limit of Navier-Stokes system with non-slip boundary condition is to deal with the well-posedness of the Prandtl system. Since there is no horizontal diffusion in the u equation of (1.1), the nonlinear term $v \partial_y u$ (which almost behaves like $\partial_x u \partial_y u$) loses one horizontal derivative in the process of energy estimate, and therefore the question of whether or not the Prandtl system with general data is well-posed in Sobolev spaces is still open. Recently, Gérard-Varet and Dormy [9] proved the ill-posedness in Sobolev spaces for the linearized Prandtl system around non-monotonic shear flows. The

nonlinear ill-posedness was also established in [11, 12] in the sense of non-Lipschitz continuity of the flow. Nevertheless, we have the following positive results for two classes of special data.

- Under a monotonic assumption on the tangential velocity of the outflow and $d = 2$, Oleinik [15] proved the local existence and uniqueness of classical solutions to (1.1). With the additional “favorable” condition on the pressure, Xin and Zhang [22] obtained the global existence of weak solutions to this system. The main idea in [15, 22] is to use Crocco transformation. We refer to [1, 13] for recent proofs and generalizations of such kind of results which is based on the direct energy method.

- For the data which is analytic in x, y variables, Sammartino and Caffisch [21] established the local well-posedness result of (1.1). Later, the analyticity in y variable was removed by Lombardo, Cannone and Sammartino in [14]. The main argument used in [21, 14] is to apply the abstract Cauchy-Kowalewska (CK) theorem. We also mention a recent well-posedness result of (1.1) for a class of data with Gevrey regularity [10].

On the other hand, Chemin, Gallagher and Paicu [5] (see also [18, 19]) proved an interesting result concerning the global well-posedness of three dimensional Navier-Stokes system with a class of “ill prepared data”, which is slowly varying in the vertical variable, namely of the form εx_3 , and the critical norm of which blows up as the small parameter goes to zero. The main idea of the proof in [5] (see also [18, 19]) is that: after a change of scale, one obtains anisotropic Navier-Stokes system which has such diffusion term as $\Delta_h + \varepsilon^3 \partial_{x_3}^2$ and anisotropic pressure gradient of the form $-(\nabla_h p, \varepsilon^2 \partial_{x_3} p)$, therefore there is a loss of regularity in the vertical variable in the classical Sobolev estimates and it is natural to work this transformed problem with initial data in the analytical spaces. However the main disadvantage of CK type argument is that it does not provide either the explicit radius of analyticity or the lifespan of the solution. The main idea to overcome those difficulties is to use the tool introduced by Chemin [4] which consists in making analytical type estimates and controlling the size of the analytic band simultaneously. One may check [4, 5, 18, 19] and the references therein concerning related results on analytic solutions to classical Navier-Stokes system.

Motivated by [4, 5, 18, 19], we are going to investigate the long time well-posedness of Prandtl system with small and analytic initial data. Since there is no horizontal diffusion in the System (1.1), this system looks like a hyperbolic one. Hence, it is natural to expect that the lifespan of the solutions to this problem with initial data of size ε should be of size $O(\varepsilon^{-1})$. Indeed, this is possible if we go through the proofs of classical CK type argument step by step. By making full use of the vertical diffusion in (1.1) as well as $\operatorname{div}_h u + \partial_y v = 0$, we shall prove that vertical diffusion slows down the decreasing of the analyticity radius so that the lifespan of the solution becomes longer.

We remark that since the y variable only lies in the upper half line, it is natural to use the L^2 framework in process of energy estimate of (1.1). However, note that this approach only gains half derivative, and to solve the well-posedness of (1.1) (see [4]), one needs one more derivative in the horizontal variables. In order to gain this additional one derivative, we will have to use the weighted Chemin-Lerner spaces introduced by Paicu and Zhang in [17].

For simplicity, here we just consider the case of uniform outflow where $U = \varepsilon \mathbf{e}$ for some unit vector $\mathbf{e} \in \mathbb{R}^{d-1}$, which implies $\nabla_h p = 0$. Let $u^s(t, y)$ be determined by

$$(1.2) \quad \begin{cases} \partial_t u^s - \partial_{yy} u^s = 0, & (t, y) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ u^s|_{y=0} = 0 \quad \text{and} \quad \lim_{y \rightarrow \infty} u^s(t, y) = \varepsilon \mathbf{e}, \\ u^s|_{t=0} = \varepsilon \chi(y) \mathbf{e}, \end{cases}$$

where $\chi(y) \in C^\infty(\mathbb{R})$, and $\chi(y) = 0$ for $y \leq 1$ and $\chi(y) = 1$ for $y \geq 2$.

By substituting $u = u^s + w$ in (1.1) and using (1.2), we write

$$(1.3) \quad \begin{cases} \partial_t w + (w + u^s) \cdot \nabla_h w - \int_0^y \operatorname{div}_h w \, dy' \partial_y w - \int_0^y \operatorname{div}_h w \, dy' \partial_y u^s - \partial_{yy} w = 0, \\ w|_{y=0} = 0 \quad \text{and} \quad \lim_{y \rightarrow \infty} w = 0, \\ w|_{t=0} = u_0 - \varepsilon \chi \mathbf{e} \stackrel{\text{def}}{=} w_0. \end{cases}$$

Our main result is stated as follows.

Theorem 1.1. *Let $\delta > 0$ and ε be a sufficiently small positive constant. Assume that w_0 satisfies*

$$(1.4) \quad \|e^{\frac{1+y^2}{s}} e^{\delta|D|} w_0\|_{\mathcal{B}^{\frac{d-1}{2},0}} \leq \varepsilon,$$

then there exists a positive time T_ε which is of size greater than $\varepsilon^{-\frac{4}{3}}$ so that (1.3) has a unique solution w which satisfies

$$(1.5) \quad e^{\Psi(t,y)} e^{\Phi(t,D)} w \in \tilde{L}_T^\infty(\mathcal{B}^{\frac{d-1}{2},0}), \quad e^{\Psi(t,y)} e^{\Phi(t,D)} \partial_y w \in \tilde{L}_T^2(\mathcal{B}^{\frac{d-1}{2},0}),$$

for any $T \leq T_\varepsilon$, and where the functions $\Psi(t, y)$, $\Phi(t, \xi)$ are determined by (2.6) and (2.7) respectively.

The definitions of the function spaces will be presented in Section 2.

Remark 1.1. *We make the following comments concerning this theorem:*

- (1) *We remark that here we require our initial data to be analytic only in the x variables. Moreover, our method to prove Theorem 1.1 also ensures the local well-posedness of (1.3) with general analytic initial data of arbitrary size.*
- (2) *In view of the result by E and Enquist [8] concerning the finite time blow-up of classical solution to (1.1), one may not expect the global existence result for (1.1) with general data without monotonicity assumption in [15]. However, whether the lifespan obtained in Theorem 1.1 is sharp is a very interesting question.*
- (3) *The condition (1.4) can be relaxed to*

$$\|e^{\rho(1+y)} e^{\delta|D|} w_0\|_{\mathcal{B}^{\frac{d-1}{2},0}} \leq \varepsilon,$$

if we take $\Psi(t, y) = \rho\left(\frac{1+y}{(1+t)^{\frac{1}{2}}}\right)$, where $\rho(z) \in W^{2,\infty}(\mathbb{R}_+)$ is a linear function of z for $z \geq 1$ and satisfies

$$\rho'(z)z \geq 2(\rho''(z) + 2\rho'(z)^2).$$

This paper is organized as follows. In the second section, we recall some basic facts on Littlewood-Paley theory and function spaces we are going to use. In the third section, we present the proof to the existence part of Theorem 1.1. In the fourth section, we complete the uniqueness part of Theorem 1.1.

Let us end this introduction by the notations we shall use in this context.

For $a \lesssim b$, we mean that there is a uniform constant C , which may be different on different lines but be independent of ε , such that $a \leq Cb$. $(a \mid b)_{L_+^2} \stackrel{\text{def}}{=} \int_{\mathbb{R}_+^d} a(x, y) \bar{b}(x, y) \, dx \, dy$ stands for the L^2 inner product of a, b on \mathbb{R}_+^d . For X a Banach space and I an interval of \mathbb{R} , we denote by $L^q(I; X)$ the set of measurable functions on I with values in X , such that $t \mapsto \|f(t)\|_X$ belongs to $L^q(I)$ and $L_+^q = L^q(\mathbb{R}^{d-1} \times \mathbb{R}_+)$. In particular, we denote by $L_T^p(L_h^q(L_v^r))$ the space

$L^p([0, T]; L^q(\mathbb{R}_x^{d-1}; L^r(\mathbb{R}_y^+)))$. Finally, we denote $\{d_k\}_{k \in \mathbb{Z}}$ to be generic elements in the sphere of $\ell^1(\mathbb{Z})$.

2. LITTLEWOOD-PALEY THEORY AND FUNCTIONAL FRAMEWORK

In the rest of this paper, we shall frequently use Littlewood-Paley decomposition in the horizontal variables x . Let us recall from [2] that

$$(2.1) \quad \Delta_k^h a = \mathcal{F}^{-1}(\varphi(2^{-k}|\xi|)\hat{a}), \quad S_k^h a = \mathcal{F}^{-1}(\chi(2^{-k}|\xi|)\hat{a}),$$

where and in all that follows, $\mathcal{F}a$ and \hat{a} always denote the partial Fourier transform of the distribution a with respect to x variables, that is, $\hat{a}(\xi, y) = \mathcal{F}_{x \rightarrow \xi}(a)(\xi, y)$, and $\chi(\tau)$, $\varphi(\tau)$ are smooth functions such that

$$\begin{aligned} \text{Supp } \varphi &\subset \left\{ \tau \in \mathbb{R} / \frac{3}{4} \leq |\tau| \leq \frac{8}{3} \right\} \quad \text{and} \quad \forall \tau > 0, \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\tau) = 1, \\ \text{Supp } \chi &\subset \left\{ \tau \in \mathbb{R} / |\tau| \leq \frac{4}{3} \right\} \quad \text{and} \quad \chi(\tau) + \sum_{j \geq 0} \varphi(2^{-j}\tau) = 1. \end{aligned}$$

Let us also recall the functional spaces we are going to use.

Definition 2.1. Let s in \mathbb{R} . For u in $\mathcal{S}'_h(\mathbb{R}_+^d)$, which means that u is in $\mathcal{S}'(\mathbb{R}_+^d)$ and satisfies $\lim_{k \rightarrow -\infty} \|S_k^h u\|_{L^\infty} = 0$, we set

$$\|u\|_{\mathcal{B}^{s,0}} \stackrel{\text{def}}{=} \left\| (2^{ks} \|\Delta_k^h u\|_{L_+^2})_k \right\|_{\ell^1(\mathbb{Z})}.$$

- For $s \leq \frac{d-1}{2}$, we define $\mathcal{B}^{s,0}(\mathbb{R}_+^d) \stackrel{\text{def}}{=} \{u \in \mathcal{S}'_h(\mathbb{R}_+^d) \mid \|u\|_{\mathcal{B}^{s,0}} < \infty\}$.
- If k is a positive integer and if $\frac{d-1}{2} + k < s \leq \frac{d+1}{2} + k$, then we define $\mathcal{B}^{s,0}(\mathbb{R}_+^d)$ as the subset of distributions u in $\mathcal{S}'_h(\mathbb{R}_+^d)$ such that $\nabla_h^\beta u$ belongs to $\mathcal{B}^{s-k,0}(\mathbb{R}_+^d)$ whenever $|\beta| = k$.

In order to obtain a better description of the regularizing effect of the transport-diffusion equation, we need to use Chemin-Lerner type spaces $\tilde{L}_T^\lambda(\mathcal{B}^{s,0}(\mathbb{R}_+^d))$.

Definition 2.2. Let $p \in [1, +\infty]$ and $T \in]0, +\infty]$. We define $\tilde{L}_T^p(\mathcal{B}^{s,0}(\mathbb{R}_+^d))$ as the completion of $C([0, T]; \mathcal{S}(\mathbb{R}_+^d))$ by the norm

$$\|a\|_{\tilde{L}_T^p(\mathcal{B}^{s,0})} \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}} 2^{ks} \left(\int_0^T \|\Delta_k^h a(t)\|_{L_+^2}^p dt \right)^{\frac{1}{p}}$$

with the usual change if $r = \infty$.

In order to overcome the difficulty that one can not use Gronwall's type argument in the framework of Chemin-Lerner space $\tilde{L}_t^2(\mathcal{B}^{s,0})$, we also need to use the weighted Chemin-Lerner norm, which was introduced by Paicu and Zhang in [17].

Definition 2.3. Let $f(t) \in L_{loc}^1(\mathbb{R}_+)$ be a nonnegative function. We define

$$(2.2) \quad \|a\|_{\tilde{L}_{t,f}^p(\mathcal{B}^{s,0})} \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}} 2^{ks} \left(\int_0^t f(t') \|\Delta_k^h a(t')\|_{L_+^2}^p dt' \right)^{\frac{1}{p}}.$$

For the convenience of the readers, we recall the following anisotropic Bernstein type lemma from [6, 16]:

Lemma 2.1. *Let \mathcal{B}_h be a ball of \mathbb{R}_h^{d-1} , and \mathcal{C}_h a ring of \mathbb{R}_h^{d-1} ; let $1 \leq p_2 \leq p_1 \leq \infty$ and $1 \leq q \leq \infty$. Then there holds:*

If the support of \hat{a} is included in $2^k \mathcal{B}_h$, then

$$\|\partial_x^\alpha a\|_{L_h^{p_1}(L_v^q)} \lesssim 2^{k(|\alpha|+(d-1)(\frac{1}{p_2}-\frac{1}{p_1}))} \|a\|_{L_h^{p_2}(L_v^q)}.$$

If the support of \hat{a} is included in $2^k \mathcal{C}_h$, then

$$\|a\|_{L_h^{p_1}(L_v^q)} \lesssim 2^{-kN} \sup_{|\alpha|=N} \|\partial_x^\alpha a\|_{L_h^{p_1}(L_v^q)}.$$

We shall constantly use the Bony's decomposition (see [3]) for the horizontal variables:

$$(2.3) \quad fg = T_f^h g + T_g^h f + R^h(f, g),$$

where

$$T_f^h g = \sum_k S_{k-1}^h f \Delta_k^h g, \quad R^h(f, g) = \sum_k \tilde{\Delta}_k^h f \Delta_k^h g$$

$$\text{with } \tilde{\Delta}_k^h f \stackrel{\text{def}}{=} \sum_{|k-k'| \leq 1} \Delta_{k'}^h f.$$

As in [4, 5, 18, 19], for any locally bounded function Φ on $\mathbb{R}^+ \times \mathbb{R}^{d-1}$, we define

$$(2.4) \quad u_\Phi(t, x, y) = \mathcal{F}_{\xi \rightarrow x}^{-1} (e^{\Phi(t, \xi)} \hat{u}(t, \xi, y)).$$

We introduce a key quantity $\theta(t)$ to describe the evolution of the analytic band of w :

$$(2.5) \quad \begin{cases} \dot{\theta}(t) = \langle t \rangle^{\frac{1}{4}} (\|e^\Psi \partial_y w_\Phi(t)\|_{\mathcal{B}^{\frac{d-1}{2}, 0}} + \|e^\Psi \partial_y u^s(t)\|_{L_v^2}), \\ \theta|_{t=0} = 0. \end{cases}$$

Here $\langle t \rangle = 1 + t$, the phase function Φ is defined by

$$(2.6) \quad \Phi(t, \xi) \stackrel{\text{def}}{=} (\delta - \lambda \theta(t)) |\xi|,$$

and the weighted function $\Psi(t, y)$ is determined by

$$(2.7) \quad \Psi(t, y) \stackrel{\text{def}}{=} \frac{1 + y^2}{8 \langle t \rangle},$$

which satisfies

$$(2.8) \quad \partial_t \Psi(t, y) + 2(\partial_y \Psi(t, y))^2 \leq 0.$$

3. THE PROOF OF THE EXISTENCE PART OF THEOREM 1.1

The general strategy to prove the existence result for a nonlinear partial differential equation is first to construct an appropriate approximate solutions, then perform uniform estimates for such approximate solution sequence, and finally pass to the limit of the approximate problem. For simplicity, here we only present the *a priori* estimates for smooth enough solutions of (1.3) in the analytical framework.

In view of (1.3), (2.4) and (2.6), it is easy to observe that w_Φ verifies

$$(3.1) \quad \begin{aligned} & \partial_t w_\Phi + \lambda \dot{\theta}(t) |D_h| w_\Phi + [w \cdot \nabla_h w]_\Phi - \left[\int_0^y \operatorname{div}_h w \, dy' \partial_y w \right]_\Phi \\ & + u^s \cdot \nabla_h w_\Phi - \int_0^y \operatorname{div}_h w_\Phi \, dy' \partial_y u^s - \partial_{yy} w_\Phi = 0, \end{aligned}$$

where $|D_h|$ denotes the Fourier multiplier with symbol $|\xi|$.

Let $\Phi(t, \xi)$ and $\Psi(t, y)$ be given by (2.6) and (2.7) respectively. By applying the dyadic operator Δ_k^h to (3.1) and then taking the L_+^2 inner product of the resulting equation with $e^{2\Psi} \Delta_k^h w_\Phi$, we find

$$\begin{aligned}
(3.2) \quad & (e^\Psi \partial_t \Delta_k^h w_\Phi \mid e^\Psi \Delta_k^h w_\Phi)_{L_+^2} + \lambda \dot{\theta} (e^\Psi |D_h| \Delta_k^h w_\Phi \mid e^\Psi \Delta_k^h w_\Phi)_{L_+^2} \\
& - (e^\Psi \partial_{yy} \Delta_k^h w_\Phi \mid e^\Psi \Delta_k^h w_\Phi)_{L_+^2} + (e^\Psi \Delta_k^h [w \cdot \nabla_h w]_\Phi \mid e^\Psi \Delta_k^h w_\Phi)_{L_+^2} \\
& - (e^\Psi \Delta_k^h [\int_0^y \operatorname{div}_h w \, dy' \partial_y w]_\Phi \mid e^\Psi \Delta_k^h w_\Phi)_{L_+^2} + (e^\Psi u^s \cdot \nabla_h \Delta_k^h w_\Phi \mid e^\Psi \Delta_k^h w_\Phi)_{L_+^2} \\
& - (e^\Psi \int_0^y \operatorname{div}_h \Delta_k^h w_\Phi \, dy' \partial_y u^s \mid e^\Psi \Delta_k^h w_\Phi)_{L_+^2} = 0.
\end{aligned}$$

In what follows, we shall always assume that $t < T^*$ with T^* being determined by

$$(3.3) \quad T^* \stackrel{\text{def}}{=} \sup \{ t > 0, \quad \theta(t) < \delta/\lambda \}.$$

So that by virtue of (2.6), for any $t < T^*$, there holds the following convex inequality

$$(3.4) \quad \Phi(t, \xi) \leq \Phi(t, \xi - \eta) + \Phi(t, \eta) \quad \text{for } \forall \xi, \eta \in \mathbb{R}^{d-1}.$$

Let us now handle term by term in (3.2).

- Estimate of $\int_0^t (e^\Psi \partial_t \Delta_k^h w_\Phi \mid e^\Psi \Delta_k^h w_\Phi)_{L_+^2} dt'$

We first get, by using integration by parts, that

$$(e^\Psi \partial_t \Delta_k^h w_\Phi \mid e^\Psi \Delta_k^h w_\Phi)_{L_+^2} = (\partial_t (e^\Psi \Delta_k^h w_\Phi) \mid e^\Psi \Delta_k^h w_\Phi)_{L_+^2} - (\partial_t \Psi (e^\Psi \Delta_k^h w_\Phi) \mid e^\Psi \Delta_k^h w_\Phi)_{L_+^2},$$

Integrating the above equality over $[0, t]$ gives rise to

$$\begin{aligned}
(3.5) \quad & \int_0^t (e^\Psi \partial_t \Delta_k^h w_\Phi \mid e^\Psi \Delta_k^h w_\Phi)_{L_+^2} dt' = \frac{1}{2} \|e^\Psi \Delta_k^h w_\Phi(t)\|_{L_+^2}^2 - \frac{1}{2} \|e^{\frac{1+y^2}{8}} \Delta_k^h e^{\delta|D|} w_0\|_{L_+^2}^2 \\
& - \int_0^t \int_{\mathbb{R}_+^d} \partial_t \Psi |e^\Psi \Delta_k^h w_\Phi(t')|^2 dx dy dt'.
\end{aligned}$$

- Estimate of $\int_0^t (e^\Psi \Delta_k^h [w \cdot \nabla_h w]_\Phi \mid e^\Psi \Delta_k^h w_\Phi)_{L_+^2} dt'$

Applying Bony's decomposition (2.3) for $w \cdot \nabla_h w$ for the x variables gives

$$w \cdot \nabla_h w = T_w^h \nabla_h w + T_{\nabla_h w}^h w + R^h(w, \nabla_h w).$$

Considering (3.4) and the support properties to the Fourier transform of the terms in $T_w^h \nabla_h w$, we write

$$\begin{aligned}
& \int_0^t |(e^\Psi \Delta_k^h [T_w^h \nabla_h w]_\Phi \mid e^\Psi \Delta_k^h w_\Phi)_{L_+^2}| dt' \\
& \lesssim \sum_{|k'-k| \leq 4} \int_0^t \|S_{k'-1}^h w_\Phi(t')\|_{L_+^\infty} \|e^\Psi \Delta_{k'}^h \nabla_h w_\Phi(t')\|_{L_+^2} \|e^\Psi \Delta_k^h w_\Phi(t')\|_{L_+^2} dt'.
\end{aligned}$$

However note that $w|_{y=0} = 0$, one has

$$\|S_{k'-1}^h w_\Phi(t')\|_{L_+^\infty} = \left\| S_{k'-1}^h \left(\int_0^y \partial_y w_\Phi(t') dy' \right) \right\|_{L_+^\infty} \leq \|\partial_y w_\Phi(t')\|_{L_v^1(L_h^\infty)},$$

whereas applying Lemma 2.1 and using (2.7) yields

$$\begin{aligned} \|\partial_y w_\Phi(t')\|_{L^1_\vee(L^\infty_h)} &\lesssim \sum_{k \in \mathbb{Z}} 2^{\left(\frac{d-1}{2}\right)k} \|\Delta_k^h \partial_y w_\Phi(t')\|_{L^1_\vee(L^2_h)} \\ &\lesssim \|e^{-\Psi(t')}\|_{L^2_\vee} \sum_{k \in \mathbb{Z}} 2^{\left(\frac{d-1}{2}\right)k} \|e^\Psi \Delta_k^h \partial_y w_\Phi(t')\|_{L^2_+} \\ &\lesssim \langle t' \rangle^{\frac{1}{4}} \|e^\Psi \partial_y w_\Phi(t')\|_{\mathcal{B}^{\frac{d-1}{2},0}}, \end{aligned}$$

which together with (2.5) ensures that

$$(3.6) \quad \|S_{k'-1}^h w_\Phi(t')\|_{L^\infty_+} \lesssim \dot{\theta}(t').$$

Whence in view of Definition 2.3 and by applying Hölder's inequality, we obtain

$$\begin{aligned} (3.7) \quad &\int_0^t |(e^\Psi \Delta_k^h [T_w^h \nabla_h w]_\Phi | e^\Psi \Delta_k^h w_\Phi)_{L^2_+}| dt' \\ &\lesssim \sum_{|k'-k| \leq 4} 2^{k'} \left(\int_0^t \dot{\theta}(t') \|e^\Psi \Delta_{k'}^h w_\Phi(t')\|_{L^2_+}^2 dt' \right)^{\frac{1}{2}} \left(\int_0^t \dot{\theta}(t') \|e^\Psi \Delta_k^h w_\Phi(t')\|_{L^2_+}^2 dt' \right)^{\frac{1}{2}} \\ &\lesssim d_k^2 2^{-(d-1)k} \|e^\Psi w_\Phi\|_{\tilde{L}^2_{t,\dot{\theta}(t)}(\mathcal{B}^{\frac{d}{2},0})}^2. \end{aligned}$$

Along the same line, it follows from Lemma 2.1 and (3.6) that

$$\|S_{k'-1}^h \nabla_h w_\Phi(t)\|_{L^\infty_+} \lesssim 2^{k'} \|S_{k'-1}^h w_\Phi(t)\|_{L^\infty_+} \lesssim 2^{k'} \dot{\theta}(t),$$

from which and

$$\begin{aligned} &\int_0^t |(e^\Psi \Delta_k^h [T_{\nabla_h w}^h w]_\Phi | e^\Psi \Delta_k^h w_\Phi)_{L^2_+}| dt' \\ &\lesssim \sum_{|k'-k| \leq 4} \int_0^t \|S_{k'-1}^h \nabla_h w_\Phi(t')\|_{L^\infty_+} \|e^\Psi \Delta_{k'}^h w_\Phi(t')\|_{L^2_+} \|e^\Psi \Delta_k^h w_\Phi(t')\|_{L^2_+} dt', \end{aligned}$$

we thus deduce by a similar derivation of (3.7) that

$$\int_0^t |(e^\Psi \Delta_k^h [T_{\nabla_h w}^h w]_\Phi | e^\Psi \Delta_k^h w_\Phi)_{L^2_+}| dt' \lesssim d_k^2 2^{-(d-1)k} \|e^\Psi w_\Phi\|_{\tilde{L}^2_{t,\dot{\theta}(t)}(\mathcal{B}^{\frac{d}{2},0})}^2.$$

Finally due to (3.4) and the support properties to the Fourier transform of the terms in $R^h(w, \nabla_h w)$, we write

$$\begin{aligned} &\int_0^t |(e^\Psi \Delta_k^h [R^h(w, \nabla_h w)]_\Phi | e^\Psi \Delta_k^h w_\Phi)_{L^2_+}| dt' \\ &\lesssim 2^{\left(\frac{d-1}{2}\right)k} \sum_{k' \geq k-3} \int_0^t \|\tilde{\Delta}_{k'}^h \nabla_h w_\Phi(t')\|_{L^\infty_\vee(L^2_h)} \|e^\Psi \Delta_{k'}^h w_\Phi(t')\|_{L^2_+} \|e^\Psi \Delta_k^h w_\Phi(t')\|_{L^2_+} dt', \end{aligned}$$

yet observing that

$$\begin{aligned} \|\tilde{\Delta}_{k'}^h \nabla_h w_\Phi(t')\|_{L^\infty_\vee(L^2_h)} &\lesssim 2^{k'} \|\tilde{\Delta}_{k'}^h w_\Phi(t')\|_{L^2_h(L^\infty_\vee)} \lesssim 2^{k'} \|\tilde{\Delta}_{k'}^h \partial_y w_\Phi(t')\|_{L^2_h(L^1_\vee)} \\ &\lesssim 2^{k'} \langle t' \rangle^{\frac{1}{4}} \|e^\Psi \tilde{\Delta}_{k'}^h \partial_y w_\Phi(t')\|_{L^2_+} \lesssim 2^{\left(\frac{3-d}{2}\right)k} \langle t' \rangle^{\frac{1}{4}} \|\partial_y w_\Phi(t')\|_{\mathcal{B}^{\frac{d-1}{2},0}} \\ &\lesssim 2^{\left(\frac{3-d}{2}\right)k} \dot{\theta}(t'). \end{aligned}$$

we find

$$\begin{aligned} & \int_0^t |(e^\Psi \Delta_k^h [R^h(w, \nabla_h w)]_\Phi | e^\Psi \Delta_k^h w_\Phi)_{L_+^2} dt' \\ & \lesssim 2^{\left(\frac{d-1}{2}\right)k} \sum_{k' \geq k-3} 2^{\left(\frac{3-d}{2}\right)k'} \left(\int_0^t \dot{\theta}(t') \|e^\Psi \Delta_{k'}^h w_\Phi(t')\|_{L_+^2}^2 dt' \right)^{\frac{1}{2}} \\ & \quad \times \left(\int_0^t \dot{\theta}(t') \|e^\Psi \Delta_k^h w_\Phi(t')\|_{L_+^2}^2 dt' \right)^{\frac{1}{2}}, \end{aligned}$$

which together with Definition 2.3 ensures that

$$\begin{aligned} (3.8) \quad & \int_0^t |(e^\Psi \Delta_k^h [R^h(w, \nabla_h w)]_\Phi | e^\Psi \Delta_k^h w_\Phi)_{L_+^2} dt' \\ & \lesssim d_k 2^{-\frac{k}{2}} \left(\sum_{k' \geq k-3} d_{k'} 2^{-\left(\frac{2d-3}{2}\right)k'} \right) \|e^\Psi w_\Phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{\frac{d}{2}, 0})}^2 \\ & \lesssim d_k^2 2^{-(d-1)k} \|e^\Psi w_\Phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{\frac{d}{2}, 0})}^2. \end{aligned}$$

In summary, we arrive at

$$(3.9) \quad \int_0^t |(e^\Psi \Delta_k^h [w \cdot \nabla_h w]_\Phi | e^\Psi \Delta_k^h w_\Phi)_{L_+^2} dt' \lesssim d_k^2 2^{-(d-1)k} \|e^\Psi w_\Phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{\frac{d}{2}, 0})}^2.$$

- Estimate of $\int_0^t (e^\Psi \Delta_k^h [\int_0^y \operatorname{div}_h w dy' \partial_y w]_\Phi | e^\Psi \Delta_k^h w_\Phi)_{L_+^2} dt'$

Once again we first get, by applying Bony's decomposition (2.3) for $\int_0^y \operatorname{div}_h w dy' \partial_y w$ in the horizontal variable x , that

$$\int_0^y \operatorname{div}_h w dy' \partial_y w = T_{\int_0^y \operatorname{div}_h w dy'}^h \partial_y w + T_{\partial_y w}^h \int_0^y \operatorname{div}_h w dy' + R^h \left(\int_0^y \operatorname{div}_h w dy', \partial_y w \right).$$

It is easy to observe that

$$\begin{aligned} & \int_0^t |(e^\Psi \Delta_k^h [T_{\int_0^y \operatorname{div}_h w dy'}^h \partial_y w]_\Phi | e^\Psi \Delta_k^h w_\Phi)_{L_+^2} dt' \\ & \lesssim \sum_{|k'-k| \leq 4} \int_0^t \|S_{k'-1}^h \left(\int_0^y \operatorname{div}_h w_\Phi(t') dy' \right)\|_{L_+^\infty} \|e^\Psi \Delta_{k'}^h \partial_y w_\Phi(t')\|_{L_+^2} \|e^\Psi \Delta_k^h w_\Phi(t')\|_{L_+^2} dt' \\ & \lesssim \sum_{|k'-k| \leq 4} 2^{-\left(\frac{d-1}{2}\right)k'} \int_0^t \|S_{k'-1}^h \left(\int_0^y \operatorname{div}_h w_\Phi(t') dy' \right)\|_{L_+^\infty} \\ & \quad \times \|e^\Psi \partial_y w_\Phi(t')\|_{\mathcal{B}^{\frac{d-1}{2}, 0}} \|e^\Psi \Delta_k^h w_\Phi(t')\|_{L_+^2} dt'. \end{aligned}$$

Then due to (2.5), one has

$$\begin{aligned} & \int_0^t |(e^\Psi \Delta_k^h [T_{\int_0^y \operatorname{div}_h w dy'}^h \partial_y w]_\Phi | e^\Psi \Delta_k^h w_\Phi)_{L_+^2} dt' \\ & \lesssim \sum_{|k'-k| \leq 4} 2^{-\left(\frac{d-1}{2}\right)k'} \left(\int_0^t \langle t' \rangle^{-\frac{1}{2}} \dot{\theta}(t') \|S_{k'-1}^h \left(\int_0^y \operatorname{div}_h w_\Phi(t') dy' \right)\|_{L_+^\infty}^2 dt' \right)^{\frac{1}{2}} \\ & \quad \times \left(\int_0^t \dot{\theta}(t') \|e^\Psi \Delta_k^h w_\Phi(t')\|_{L_+^2}^2 dt' \right)^{\frac{1}{2}}. \end{aligned}$$

However in view of Definition 2.3, by applying Lemma 2.1, one has

$$\begin{aligned}
& \left(\int_0^t \langle t' \rangle^{-\frac{1}{2}} \dot{\theta}(t') \|S_{k'-1}^h \int_0^y \operatorname{div}_h w_\Phi(t') dy'\|_{L_+^\infty}^2 dt' \right)^{\frac{1}{2}} \\
& \lesssim \sum_{\ell \leq k'-2} 2^{\left(\frac{d+1}{2}\right)\ell} \left(\int_0^t \langle t' \rangle^{-\frac{1}{2}} \dot{\theta}(t') \|\Delta_\ell^h w_\Phi(t')\|_{L_v^1(L_h^2)}^2 dt' \right)^{\frac{1}{2}} \\
& \lesssim \sum_{\ell \leq k'-2} 2^{\left(\frac{d+1}{2}\right)\ell} \left(\int_0^t \dot{\theta}(t') \|e^\Psi \Delta_\ell^h w_\Phi(t')\|_{L_+^2}^2 dt' \right)^{\frac{1}{2}} \\
& \lesssim d_{k'} 2^{\frac{k'}{2}} \|e^\Psi w_\Phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{\frac{d}{2},0})}.
\end{aligned}$$

Hence we obtain

$$\int_0^t |(e^\Psi \Delta_k^h [T_{\int_0^y \operatorname{div}_h w dy'} \partial_y w]_\Phi | e^\Psi \Delta_k^h w_\Phi)_{L_+^2}| dt' \lesssim d_k^2 2^{-(d-1)k} \|e^\Psi w_\Phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{\frac{d}{2},0})}^2.$$

By the same manner, we write

$$\begin{aligned}
& \int_0^t |(e^\Psi \Delta_k^h [T_{\partial_y w} \int_0^y \operatorname{div}_h w dy']_\Phi | e^\Psi \Delta_k^h w_\Phi)_{L_+^2}| dt' \\
& \lesssim \sum_{|k'-k| \leq 4} \int_0^t \|e^\Psi S_{k'-1}^h(\partial_y w_\Phi(t'))\|_{L_v^2(L_h^\infty)} \\
& \quad \times \|\Delta_{k'}^h \int_0^y \operatorname{div}_h w_\Phi(t') dy'\|_{L_v^\infty(L_h^2)} \|e^\Psi \Delta_k^h w_\Phi(t')\|_{L_+^2} dt' \\
& \lesssim \sum_{|k'-k| \leq 4} 2^{k'} \int_0^t \|e^\Psi \partial_y w_\Phi(t')\|_{\mathcal{B}^{\frac{d-1}{2},0}} \|\Delta_{k'}^h w_\Phi(t')\|_{L_v^1(L_h^2)} \|e^\Psi \Delta_k^h w_\Phi(t')\|_{L_+^2} dt' \\
& \lesssim \sum_{|k'-k| \leq 4} 2^{k'} \int_0^t \dot{\theta}(t') \|e^\Psi \Delta_{k'}^h w_\Phi(t')\|_{L_+^2} \|e^\Psi \Delta_k^h w_\Phi(t')\|_{L_+^2} dt',
\end{aligned}$$

which along with a similar derivation of (3.7) leads to

$$\int_0^t |(e^\Psi \Delta_k^h [T_{\partial_y w} \int_0^y \operatorname{div}_h w dy']_\Phi | e^\Psi \Delta_k^h w_\Phi)_{L_+^2}| dt' \lesssim d_k^2 2^{-(d-1)k} \|e^\Psi w_\Phi\|_{\tilde{L}_{t,\dot{\theta}(t)}^2(\mathcal{B}^{\frac{d}{2},0})}^2.$$

Finally, it follows from Lemma 2.1 that

$$\begin{aligned}
& \int_0^t |(e^\Psi \Delta_k^h [R^h(\int_0^y \operatorname{div}_h w dy', \partial_y w)]_\Phi | e^\Psi \Delta_k^h w_\Phi)_{L_+^2}| dt' \\
& \lesssim 2^{\left(\frac{d-1}{2}\right)k} \sum_{k' \geq k-3} \int_0^t \|\Delta_{k'}^h(\int_0^y \operatorname{div}_h w_\Phi(t') dy')\|_{L_v^\infty(L_h^2)} \\
& \quad \times \|e^\Psi \tilde{\Delta}_{k'}^h \partial_y w_\Phi(t')\|_{L_+^2} \|e^\Psi \Delta_k^h w_\Phi(t')\|_{L_+^2} dt'
\end{aligned}$$

yet since

$$\begin{aligned}
\|\Delta_{k'}^h(\int_0^y \operatorname{div}_h w_\Phi(t') dy')\|_{L_v^\infty(L_h^2)} & \lesssim 2^{k'} \|\Delta_{k'}^h w_\Phi(t')\|_{L_v^1(L_h^2)} \\
& \lesssim 2^{k'} \langle t' \rangle^{\frac{1}{4}} \|e^\Psi \Delta_{k'}^h w_\Phi(t')\|_{L_+^2},
\end{aligned}$$

we deduce by a similar derivation of (3.8) that

$$\begin{aligned}
& \int_0^t |(e^\Psi \Delta_k^h [R^h(\int_0^y \operatorname{div}_h w dy', \partial_y w)]_\Phi | e^\Psi \Delta_k^h w_\Phi)_{L_+^2} dt' \\
& \lesssim 2^{(\frac{d-1}{2})k} \sum_{k' \geq k-3} 2^{(\frac{3-d}{2})k'} \int_0^t \langle t' \rangle^{\frac{1}{4}} \|\partial_y w_\Phi(t')\|_{\mathcal{B}^{\frac{d-1}{2},0}} \|e^\Psi \Delta_{k'}^h w_\Phi(t')\|_{L_+^2} \|e^\Psi \Delta_k^h w_\Phi(t')\|_{L_+^2} dt' \\
& \lesssim 2^{(\frac{d-1}{2})k} \sum_{k' \geq k-3} 2^{(\frac{3-d}{2})k'} \left(\int_0^t \dot{\theta}(t') \|e^\Psi \Delta_{k'}^h w_\Phi(t')\|_{L_+^2}^2 dt' \right)^{\frac{1}{2}} \left(\int_0^t \dot{\theta}(t') \|e^\Psi \Delta_k^h w_\Phi(t')\|_{L_+^2}^2 dt' \right)^{\frac{1}{2}} \\
& \lesssim d_k^2 2^{-(d-1)k} \|e^\Psi w_\Phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{\frac{d}{2},0})}^2.
\end{aligned}$$

As a consequence, we achieve

$$(3.10) \quad \int_0^t |(e^\Psi \Delta_k^h [\int_0^y \operatorname{div}_h w dy' \partial_y w]_\Phi | e^\Psi \Delta_k^h w_\Phi)_{L_+^2} dt' \lesssim d_k^2 2^{-(d-1)k} \|e^\Psi w_\Phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{\frac{d}{2},0})}^2.$$

- Estimate of $\int_0^t (e^\Psi \int_0^y \Delta_k^h \operatorname{div}_h w_\Phi dy' \partial_y u^s | e^\Psi \Delta_k^h w_\Phi)_{L_+^2} dt'$

By virtue of Definition 2.3 and (2.5), we get

$$\begin{aligned}
& \int_0^t |(e^\Psi \int_0^y \Delta_k^h \operatorname{div}_h w_\Phi dy' \partial_y u^s | e^\Psi \Delta_k^h w_\Phi)_{L_+^2} dt' \\
& \lesssim 2^k \int_0^t \|e^\Psi \partial_y u^s(t')\|_{L_v^2} \|\Delta_k^h w_\Phi(t')\|_{L_v^1(L_h^2)} \|e^\Psi \Delta_k^h w_\Phi(t')\|_{L_+^2} dt' \\
& \lesssim 2^k \int_0^t \dot{\theta}(t') \|e^\Psi \Delta_k^h w_\Phi(t')\|_{L_+^2} \|e^\Psi \Delta_k^h w_\Phi(t')\|_{L_+^2} dt',
\end{aligned}$$

from which, a similar derivation of (3.7) gives rise to

$$(3.11) \quad \int_0^t |(e^\Psi \int_0^y \partial_x \Delta_k^h w_\Phi dy' \partial_y u^s | e^\Psi \Delta_k^h w_\Phi)_{L_+^2} dt' \lesssim d_k^2 2^{-(d-1)k} \|e^\Psi w_\Phi\|_{\tilde{L}_{t,\theta(t)}^2(\mathcal{B}^{\frac{d}{2},0})}^2.$$

- Estimate of $-\int_0^t (e^\Psi \partial_{yy} \Delta_k^h w_\Phi | e^\Psi \Delta_k^h w_\Phi)_{L_+^2} dt'$

We get, by using integration by parts, that

$$\begin{aligned}
& -\int_0^t (e^\Psi \partial_{yy} \Delta_k^h w_\Phi | e^\Psi \Delta_k^h w_\Phi)_{L_+^2} dt' \\
(3.12) \quad & = \|e^\Psi \Delta_k^h \partial_y w_\Phi\|_{L_t^2(L_+^2)}^2 + 2 \int_0^t \int_{\mathbb{R}_+^2} \partial_y \Psi e^{2\Psi} \Delta_k^h w_\Phi(t') \Delta_k^h \partial_y w_\Phi(t') dx dy dt' \\
& \geq \frac{1}{2} \|e^\Psi \Delta_k^h \partial_y w_\Phi\|_{L_t^2(L_+^2)}^2 - 2 \int_0^t \int_{\mathbb{R}_+^d} (\partial_y \Psi)^2 |e^\Psi \Delta_k^h w_\Phi(t')|^2 dt'.
\end{aligned}$$

Now we are in a position to complete the existence part of Theorem 1.1:

Proof of the existence part of Theorem 1.1. It is easy to observe that

$$(e^\Psi u^s \cdot \nabla_h \Delta_k^h w_\Phi | e^\Psi \Delta_k^h w_\Phi)_{L_+^2} = 0,$$

and

$$\lambda \dot{\theta}(t) (e^\Psi |D_h| \Delta_k^h w_\Phi | e^\Psi \Delta_k^h w_\Phi)_{L_+^2} \geq c \lambda \dot{\theta}(t) 2^k \|e^\Psi \Delta_k^h w_\Phi(t)\|_{L_+^2}^2.$$

Therefore in view of (2.8), by integrating (3.2) over $[0, t]$ and by resuming the Estimates (3.5), (3.9), (3.10), (3.11) and (3.12) into the resulting inequality, we conclude

$$\begin{aligned} \|e^\Psi \Delta_k^h w_\Phi\|_{L_t^\infty(L_+^2)}^2 + c \lambda 2^k \int_0^t \dot{\theta}(t') \|\Delta_k^h w_\Phi(t')\|_{L_+^2}^2 dt' + \|e^\Psi \Delta_k^h \partial_y w_\Phi\|_{L_t^2(L_+^2)}^2 \\ \leq \|e^{\frac{1+y^2}{8}} \Delta_k^h e^{\delta|D|} w_0\|_{L_+^2}^2 + C d_k^2 2^{-(d-1)k} \|e^\Psi w_\Phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{\frac{d}{2}, 0})}^2. \end{aligned}$$

Taking square root of the above inequality and multiplying the resulting inequality by $2^{\left(\frac{d-1}{2}\right)k}$ and summing over $k \in \mathbb{Z}$, we find for any $t \leq T^*$

$$\begin{aligned} \|e^\Psi w_\Phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{d-1}{2}, 0})} + c \sqrt{\lambda} \|e^\Psi w_\Phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{\frac{d}{2}, 0})} + \|e^\Psi \partial_y w_\Phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{d-1}{2}, 0})} \\ \leq \|e^{\frac{1+y^2}{8}} e^{\delta|D_x|} w_0\|_{\mathcal{B}^{\frac{d-1}{2}, 0}} + \sqrt{C} \|e^\Psi w_\Phi\|_{\tilde{L}_{t, \dot{\theta}(t)}^2(\mathcal{B}^{\frac{d}{2}, 0})}. \end{aligned} \quad (3.13)$$

Taking λ to be a large enough positive constant so that $c^2 \lambda \geq C$ in (3.13) gives rise to

$$\|e^\Psi w_\Phi\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{d-1}{2}, 0})} + \|e^\Psi \partial_y w_\Phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{d-1}{2}, 0})} \leq C \|e^{\frac{1+y^2}{8}} e^{\delta|D|} w_0\|_{\mathcal{B}^{\frac{d-1}{2}, 0}}. \quad (3.14)$$

Hence in view of (2.5), we infer from Lemma 3.1 below that

$$\begin{aligned} \theta(t) &\leq \int_0^t \langle t' \rangle^{\frac{1}{4}} \|e^\Psi \partial_y w_\Phi(t')\|_{\mathcal{B}^{\frac{d-1}{2}, 0}} dt' + \int_0^t \langle t' \rangle^{\frac{1}{4}} \|e^\Psi \partial_y u^s(t')\|_{L_v^2} dt' \\ &\leq C \langle t \rangle^{\frac{3}{4}} (\|e^\Psi \partial_y w_\Phi\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{d-1}{2}, 0})} + \|e^\Psi \partial_y u^s\|_{L_t^2(L_v^2)}) \\ &\leq C \langle t \rangle^{\frac{3}{4}} \left(\|e^{\frac{1+y^2}{8}} e^{\delta|D|} w_0\|_{\mathcal{B}^{\frac{d-1}{2}, 0}} + \varepsilon \right). \end{aligned} \quad (3.15)$$

In particular, under the assumption of (1.4), (3.15) ensures that

$$\sup_{t \in [0, \tau_\varepsilon^*]} \theta(t) \leq \frac{\delta}{2\lambda} \quad \text{for } \tau_\varepsilon^* \stackrel{\text{def}}{=} \left(\frac{\delta}{4\lambda C \varepsilon} \right)^{\frac{4}{3}} - 1.$$

Therefore in view of (3.3), this ensures that $T^* \geq \tau_\varepsilon^*$, and (3.14) implies (1.5). This completes the existence part of Theorem 1.1. \square

It remains to prove the following lemma.

Lemma 3.1. *Let u^s be the global solution of (1.2). Then one has*

$$\int_0^t \|e^\Psi \partial_y u^s(t')\|_{L_v^2}^2 dt' \leq C \varepsilon^2 \quad (3.16)$$

for any $t \geq 0$.

Proof. Indeed, it is easy to observe from (1.2) that

$$u^s(t, y) = \frac{\varepsilon}{2\sqrt{\pi}t} \int_0^\infty \left(e^{-\frac{(y-y')^2}{4t}} - e^{-\frac{(y+y')^2}{4t}} \right) \chi(y') dy' \mathbf{e}.$$

Due to the choice of χ , taking derivative with respect to y gives

$$\begin{aligned}
\partial_y u^s(t, y) &= \frac{\varepsilon}{2\sqrt{\pi t}} \int_0^\infty \left(e^{-\frac{(y-y')^2}{4t}} + e^{-\frac{(y+y')^2}{4t}} \right) \partial_y \chi(y') dy' \mathbf{e} \\
&= \frac{\varepsilon}{4\sqrt{\pi t}} \int_{-\infty}^{+\infty} \left(e^{-\frac{(y-y')^2}{4t}} + e^{-\frac{(y+y')^2}{4t}} \right) \partial_y \chi(y') dy' \mathbf{e} \\
&= \partial_y \frac{\varepsilon}{4\sqrt{\pi t}} \int_{-\infty}^{+\infty} \left(e^{-\frac{(y-y')^2}{4t}} - e^{-\frac{(y+y')^2}{4t}} \right) \chi_1(y') dy' \mathbf{e} \\
&\stackrel{\text{def}}{=} \frac{\varepsilon}{2} \partial_y (u_+^s(t, y) + u_-^s(t, y)),
\end{aligned}$$

for some $\chi_1(y) \in C_c^\infty(\mathbb{R})$.

Notice that $u_\pm^s(t, y)$ verify

$$(3.17) \quad \begin{cases} \partial_t u_\pm^s(t, y) - \partial_{yy} u_\pm^s(t, y) = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}, \\ u_\pm^s|_{t=0} = \chi_1(\pm y), \end{cases}$$

Taking L^2 inner product of (3.17) with $e^{2\Psi} u_\pm^s$ and using integration by parts, we obtain

$$\begin{aligned}
0 &= (e^\Psi \partial_t u_\pm^s \mid e^\Psi u_\pm^s)_{L^2} - (e^\Psi \partial_{yy} u_\pm^s(t, y) \mid e^\Psi u_\pm^s)_{L^2} \\
&= \frac{1}{2} \frac{d}{dt} \|e^\Psi u_\pm^s(t)\|_{L^2}^2 + \|e^\Psi \partial_y u_\pm^s(t)\|_{L^2}^2 \\
&\quad - \int_{\mathbb{R}} \partial_t \Psi |e^\Psi u_\pm^s(t)|^2 dy + 2 \int_{\mathbb{R}} \partial_y \Psi e^\Psi \partial_y u_\pm^s e^\Psi u_\pm^s(t) dy,
\end{aligned}$$

which together (2.8) ensures that

$$\begin{aligned}
&\frac{1}{2} \left(\|e^\Psi u_\pm^s(t)\|_{L^2}^2 + \|e^\Psi \partial_y u_\pm^s\|_{L_t^2(L^2)}^2 \right) \\
&\leq \|e^{\frac{1+y^2}{8}} \chi_1(\pm y)\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}} (\partial_t \Psi - 2(\partial_y \Psi)^2) |e^\Psi u_\pm^s(t')|^2 dy dt' \\
&\leq \|e^{\frac{1+y^2}{8}} \chi_1(\pm y)\|_{L^2}^2.
\end{aligned}$$

And hence (3.16) follows. \square

4. THE PROOF OF THE UNIQUENESS PART OF THEOREM 1.1

This section is devoted to the proof of the uniqueness part of Theorem 1.1. Let w^1 and w^2 be two solutions of (1.3) obtained by Theorem 1.1. We denote $W \stackrel{\text{def}}{=} w^1 - w^2$. Then in view of (1.3), W verifies

$$(4.1) \quad \begin{cases} \partial_t W + u^s \cdot \nabla_h W - \int_0^y \text{div}_h \cdot W dy' \partial_y u^s - \partial_{yy} W + F = 0, \\ W|_{y=0} = 0, \quad \lim_{y \rightarrow \infty} W = 0, \\ W|_{t=0} = 0, \end{cases}$$

where

$$F = w^1 \cdot \nabla_h W + W \cdot \nabla_h w^2 - \int_0^y \text{div}_h w^1 dy' \partial_y W - \int_0^y \text{div}_h W dy' \partial_y w^2.$$

Let $\theta^i(t)$, $i = 1, 2$, be determined respectively by

$$(4.2) \quad \begin{cases} \dot{\theta}^i(t) = \langle t \rangle^{\frac{1}{4}} (\|e^\Psi \partial_y w_\Phi^i(t)\|_{\mathcal{B}^{\frac{d-1}{2},0}} + \|e^\Psi \partial_y u^s(t)\|_{L_t^2}), \\ \theta^i|_{t=0} = 0, \end{cases}$$

and the phase function $\tilde{\Phi}$ is defined by

$$(4.3) \quad \begin{aligned} \Theta(t) &\stackrel{\text{def}}{=} (\theta^1 + \theta^2)(t), \quad \tilde{\Phi}(t, \xi) \stackrel{\text{def}}{=} \left(\frac{\delta}{2} - \lambda\Theta(t)\right)|\xi| \quad \text{and} \\ \Phi^i(t, \xi) &\stackrel{\text{def}}{=} (\delta - \lambda\theta^i(t))|\xi|, \quad \text{for } i = 1, 2. \end{aligned}$$

In what follows, we shall always take t so small that

$$\frac{\delta}{2} - \lambda\Theta(t) \geq 0,$$

so that there holds

$$\tilde{\Phi}(t, \xi) \leq \tilde{\Phi}(t, \xi - \eta) + \tilde{\Phi}(t, \eta) \quad \text{for } \forall \xi, \eta \in \mathbb{R}^{d-1}.$$

Since $\tilde{\Phi}(t) \leq \min(\Phi_1(t), \Phi_2(t))$, we have for $i = 1, 2$,

$$(4.4) \quad \begin{aligned} \|e^\Psi w_\Phi^i\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{d-1}{2},0})} &\leq \|e^\Psi w_{\Phi^i}^i\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{d-1}{2},0})} \quad \text{and} \\ \|e^\Psi \partial_y w_\Phi^i\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{d-1}{2},0})} &\leq \|e^\Psi \partial_y w_{\Phi^i}^i\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{d-1}{2},0})}. \end{aligned}$$

Furthermore, it follows from Definition 2.2 and (2.4) that

$$\|e^\Psi w_\Phi^i\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{d+1}{2},0})} = \sum_{k \in \mathbb{Z}} 2^{\left(\frac{k+1}{2}\right)k} \|e^\Psi \Delta_k^h w_\Phi^i\|_{L_t^\infty(L_+^2)},$$

and for each time t , one has

$$\begin{aligned} 2^k \|e^\Psi \Delta_k^h w_\Phi^i(t)\|_{L_+^2}^2 &= 2^k \int_0^\infty \int_{\mathbb{R}^{d-1}} e^{2\Psi} \varphi^2(2^{-k}|\xi|) e^{2\tilde{\Phi}(t,\xi)} |\widehat{w^i}(t, \xi, y)|^2 d\xi dy \\ &\leq 2 \int_0^\infty \int_{\mathbb{R}^{d-1}} e^{2\Psi} \varphi^2(2^{-k}|\xi|) e^{2\Phi^i(t,\xi)} |\xi| e^{-\frac{\delta}{2}|\xi|} |\widehat{w^i}(t, \xi, y)|^2 d\xi dy \\ &\leq C_\delta \int_0^\infty \int_{\mathbb{R}^{d-1}} e^{2\Psi} \varphi^2(2^{-k}|\xi|) e^{2\Phi^i(t,\xi)} |\widehat{w^i}(t, \xi, y)|^2 d\xi dy \\ &= C_\delta \|e^\Psi \Delta_k^h w_{\Phi^i}^i(t)\|_{L_+^2}^2, \end{aligned}$$

which yields

$$(4.5) \quad \|e^\Psi w_\Phi^i\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{d+1}{2},0})} \leq C_\delta \|e^\Psi w_{\Phi^i}^i\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{d-1}{2},0})}.$$

In fact, this inequality (4.5) motivates us to introduce the phase function $\tilde{\Phi}(t, \xi)$ in (4.3).

For $\tilde{\Phi}(t, \xi)$ given by (4.3), in view of (4.1), one has

$$\partial_t W_{\tilde{\Phi}} + \lambda \dot{\Theta}(t) |D_h| W_{\tilde{\Phi}} + u^s \cdot \nabla_h W_{\tilde{\Phi}} - \int_0^y \operatorname{div}_h W_{\tilde{\Phi}} dy' \partial_y u^s - \partial_{yy} W_{\tilde{\Phi}} + F_{\tilde{\Phi}} = 0.$$

Then for $\Psi(t, y)$ given by (2.6), by applying Δ_k^h to the above equation and then taking L^2 inner product of the resulting equation with $e^{2\Psi}\Delta_k^h W_{\tilde{\Phi}}$, we obtain

$$\begin{aligned} & (e^\Psi \Delta_k^h \partial_t W_{\tilde{\Phi}} \mid e^\Psi \Delta_k^h W_{\tilde{\Phi}})_{L_+^2} + \lambda \dot{\Theta} (e^\Psi |D_h| \Delta_k^h W_{\tilde{\Phi}} \mid \Delta_k^h W_{\tilde{\Phi}})_{L_+^2} \\ & + (e^\Psi u^s \cdot \nabla_h \Delta_k^h W_{\tilde{\Phi}} \mid e^\Psi \Delta_k^h W_{\tilde{\Phi}})_{L_+^2} - (e^\Psi \Delta_k^h \partial_{yy} W_{\tilde{\Phi}} \mid e^\Psi \Delta_k^h W_{\tilde{\Phi}})_{L_+^2} \\ & - (e^\Psi \int_0^y \Delta_k^h \operatorname{div}_h W_{\tilde{\Phi}} dy' \partial_y u^s \mid e^\Psi \Delta_k^h W_{\tilde{\Phi}})_{L_+^2} + (e^\Psi \Delta_k^h F_{\tilde{\Phi}} \mid e^\Psi \Delta_k^h W_{\tilde{\Phi}})_{L_+^2}, \end{aligned}$$

from which and similar derivations of (3.5), (3.12), we infer

$$\begin{aligned} & \|e^\Psi \Delta_k^h W_{\tilde{\Phi}}\|_{L_t^\infty(L_+^2)}^2 + c\lambda 2^k \int_0^t \dot{\Theta}(t') \|e^\Psi \Delta_k^h W_{\tilde{\Phi}}(t')\|_{L_+^2}^2 dt' + \|e^\Psi \Delta_k^h \partial_y W_{\tilde{\Phi}}\|_{L_t^2(L_+^2)}^2 \\ (4.6) \quad & \leq \int_0^t \left((e^\Psi \int_0^y \Delta_k^h \operatorname{div}_h W_{\tilde{\Phi}} dy' \partial_y u^s \mid e^\Psi \Delta_k^h W_{\tilde{\Phi}})_{L_+^2} \right. \\ & \quad \left. - (e^\Psi \Delta_k^h F_{\tilde{\Phi}} \mid e^\Psi \Delta_k^h W_{\tilde{\Phi}})_{L_+^2} \right)(t') dt'. \end{aligned}$$

In what follows, we handle term by term above.

- Estimate of $\int_0^t (e^\Psi \Delta_k^h (w^1 \cdot \nabla_h W)_{\tilde{\Phi}} \mid e^\Psi \Delta_k^h W_{\tilde{\Phi}})_{L_+^2} dt'$

We first get, by using Bony's decomposition (2.3) for $w^1 \cdot \nabla_h W$ for the horizontal variables, that

$$w^1 \cdot \nabla_h W = T_{w^1}^h \nabla_h W + T_{\nabla_h W}^h w^1 + R^h(w^1, \nabla_h W).$$

By using a similar derivation of (3.6) and (3.7), we find

$$\|e^\Psi \Delta_k^h [T_{w^1}^h \nabla_h W]_{\tilde{\Phi}}(t)\|_{L_+^2} \lesssim 2^k \sum_{|k'-k| \leq 4} \langle t \rangle^{\frac{1}{4}} \|e^\Psi \partial_y w_{\tilde{\Phi}}^1(t)\|_{\mathcal{B}^{\frac{d-1}{2}, 0}} \|e^\Psi \Delta_{k'}^h W_{\tilde{\Phi}}(t)\|_{L_+^2},$$

which together with (4.3) and (4.4) implies that

$$\begin{aligned} & \int_0^t |(e^\Psi \Delta_k^h [T_{w^1}^h \nabla_h W]_{\tilde{\Phi}} \mid e^\Psi \Delta_k^h W_{\tilde{\Phi}})_{L_+^2}| dt' \\ (4.7) \quad & \lesssim 2^k \sum_{|k'-k| \leq 4} \int_0^t \dot{\Theta}(t') \|e^\Psi \Delta_{k'}^h W_{\tilde{\Phi}}(t')\|_{L_+^2} \|e^\Psi \Delta_k^h W_{\tilde{\Phi}}(t')\|_{L_+^2} dt' \\ & \lesssim 2^k \sum_{|k'-k| \leq 4} \left(\int_0^t \dot{\Theta}(t') \|e^\Psi \Delta_{k'}^h W_{\tilde{\Phi}}(t')\|_{L_+^2}^2 dt' \right)^{\frac{1}{2}} \left(\int_0^t \dot{\Theta}(t') \|e^\Psi \Delta_k^h W_{\tilde{\Phi}}(t')\|_{L_+^2}^2 dt' \right)^{\frac{1}{2}} \\ & \lesssim d_k^2 2^{-(d-1)k} \|e^\Psi W_{\tilde{\Phi}}\|_{\tilde{L}_{t, \dot{\Theta}}^2(\mathcal{B}^{\frac{d}{2}, 0})}^2. \end{aligned}$$

Note that according to (4.2) and (4.3), we get, by a similar derivation of (3.6), that

$$\begin{aligned} & \|\Delta_k^h w_{\tilde{\Phi}}^1(t)\|_{L_v^\infty(L_h^2)} \lesssim \langle t \rangle^{\frac{1}{4}} \|e^\Psi \Delta_k^h \partial_y w_{\tilde{\Phi}}^1(t)\|_{L_+^2} \\ (4.8) \quad & \lesssim 2^{-\left(\frac{d-1}{2}\right)k} \langle t \rangle^{\frac{1}{4}} \|e^\Psi \partial_y w_{\tilde{\Phi}}^1(t)\|_{\mathcal{B}^{\frac{d-1}{2}, 0}} \lesssim 2^{-\left(\frac{d-1}{2}\right)k} \dot{\Theta}(t), \end{aligned}$$

while it follows from Lemma 2.1 that

$$\begin{aligned}
& \left(\int_0^t \dot{\Theta}(t') \|e^\Psi S_{k'-1}^h \nabla_h W_{\tilde{\Phi}}(t')\|_{L_v^2(L_h^\infty)}^2 dt' \right)^{\frac{1}{2}} \\
& \lesssim \sum_{\ell \leq k'-2} 2^{\left(\frac{d+1}{2}\right)k'} \left(\int_0^t \dot{\Theta}(t') \|e^\Psi \Delta_\ell^h W_{\tilde{\Phi}}(t')\|_{L_+^2}^2 dt' \right)^{\frac{1}{2}} \\
& \lesssim d_k 2^{\frac{k'}{2}} \|e^\Psi W_{\tilde{\Phi}}\|_{\tilde{L}_{t,\Theta}^2(\mathcal{B}^{\frac{d}{2},0})}.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
& \int_0^t |(e^\Psi \Delta_k^h [T_{\nabla_h}^h W w^1]_{\tilde{\Phi}} | e^\Psi \Delta_k^h W_{\tilde{\Phi}})_{L_+^2}| dt' \\
& \lesssim \sum_{|k'-k| \leq 4} \int_0^t \|e^\Psi S_{k'-1}^h \nabla_h W_{\tilde{\Phi}}(t')\|_{L_v^2(L_h^\infty)} \|\Delta_{k'}^h w_{\tilde{\Phi}}^1(t')\|_{L_v^\infty(L_h^2)} \|e^\Psi \Delta_k^h W_{\tilde{\Phi}}(t')\|_{L_+^2} dt' \\
& \lesssim \sum_{|k'-k| \leq 4} 2^{\left(\frac{1-d}{2}\right)k'} \int_0^t \dot{\Theta}(t') \|e^\Psi S_{k'-1}^h \nabla_h W_{\tilde{\Phi}}(t')\|_{L_v^2(L_h^\infty)} \|e^\Psi \Delta_k^h W_{\tilde{\Phi}}(t')\|_{L_+^2} dt' \\
& \lesssim \sum_{|k'-k| \leq 4} 2^{\left(\frac{1-d}{2}\right)k'} \left(\int_0^t \dot{\Theta}(t') \|e^\Psi S_{k'-1}^h \nabla_h W_{\tilde{\Phi}}(t')\|_{L_v^2(L_h^\infty)}^2 dt' \right)^{\frac{1}{2}} \\
& \quad \times \left(\int_0^t \dot{\Theta}(t') \|e^\Psi \Delta_k^h W_{\tilde{\Phi}}(t')\|_{L_+^2}^2 dt' \right)^{\frac{1}{2}} \\
& \lesssim d_k^2 2^{-(d-1)k} \|e^\Psi W_{\tilde{\Phi}}\|_{\tilde{L}_{t,\Theta}^2(\mathcal{B}^{\frac{d}{2},0})}^2.
\end{aligned}$$

While by applying Lemma 2.1 and (4.8), we write

$$\begin{aligned}
& \int_0^t |(e^\Psi \Delta_k^h [R^h(w^1, \nabla_h W)]_{\tilde{\Phi}} | e^\Psi \Delta_k^h W_{\tilde{\Phi}})_{L_+^2}| dt' \\
& \lesssim 2^{\left(\frac{d-1}{2}\right)k} \sum_{k' \geq k-3} \int_0^t \|e^\Psi \Delta_{k'}^h \nabla_h W_{\tilde{\Phi}}(t')\|_{L_+^2} \|\tilde{\Delta}_{k'}^h w_{\tilde{\Phi}}^1(t')\|_{L_v^\infty(L_h^2)} \|e^\Psi \Delta_k^h W_{\tilde{\Phi}}(t')\|_{L_+^2} dt' \\
& \lesssim 2^{\left(\frac{d-1}{2}\right)k} \sum_{k' \geq k-3} 2^{\left(\frac{3-d}{2}\right)k'} \int_0^t \dot{\Theta}(t') \|e^\Psi \Delta_{k'}^h W_{\tilde{\Phi}}(t')\|_{L_+^2} \|e^\Psi \Delta_k^h W_{\tilde{\Phi}}(t')\|_{L_+^2} dt',
\end{aligned}$$

which together with Definition 2.3 implies

$$\begin{aligned}
& \int_0^t |(e^\Psi \Delta_k^h [R^h(w^1, \nabla_h W)]_{\tilde{\Phi}} | e^\Psi \Delta_k^h W_{\tilde{\Phi}})_{L_+^2}| dt' \\
(4.9) \quad & \lesssim d_k 2^{-\frac{k}{2}} \left(\sum_{k' \geq k-3} d_{k'} 2^{\left(\frac{3-2d}{2}\right)k'} \right) \|e^\Psi W_{\tilde{\Phi}}\|_{\tilde{L}_{t,\Theta}^2(\mathcal{B}^{\frac{d}{2},0})}^2 \\
& \lesssim d_k^2 2^{-(d-1)k} \|e^\Psi W_{\tilde{\Phi}}\|_{\tilde{L}_{t,\Theta}^2(\mathcal{B}^{\frac{d}{2},0})}^2.
\end{aligned}$$

Therefore, we obtain

$$(4.10) \quad \int_0^t |(e^\Psi \Delta_k^h (w^1 \cdot \nabla_h W)_{\tilde{\Phi}} | e^\Psi \Delta_k^h W_{\tilde{\Phi}})_{L_+^2}| dt' \lesssim d_k^2 2^{-(d-1)k} \|e^\Psi W_{\tilde{\Phi}}\|_{\tilde{L}_{t,\Theta}^2(\mathcal{B}^{\frac{d}{2},0})}^2.$$

- Estimate of $\int_0^t (e^\Psi \Delta_k^h (W \cdot \nabla_h w^2)_{\tilde{\Phi}} | e^\Psi \Delta_k^h W_{\tilde{\Phi}})_{L_+^2} dt'$

By using Bony's decomposition (2.3) for $W \cdot \nabla_h w^2$ for the horizontal variables, we get

$$W \cdot \nabla_h w^2 = T_W^h \nabla_h w^2 + T_{\nabla_h w^2}^h W + R^h(W, \nabla_h w^2).$$

Along the same line to the derivation of (4.7) and (4.9), we find

$$\int_0^t |(e^\Psi \Delta_k^h [T_{\nabla_h w^2}^h W]_{\tilde{\Phi}} | e^\Psi \Delta_k^h W_{\tilde{\Phi}})_{L_+^2}| dt' \lesssim d_k^2 2^{-(d-1)k} \|e^\Psi W_{\tilde{\Phi}}\|_{\tilde{L}_{t,\Theta}^2(\mathcal{B}^{\frac{d}{2},0})}^2,$$

and

$$\int_0^t |(e^\Psi \Delta_k^h [R^h(W, \nabla_h w^2)]_{\tilde{\Phi}} | e^\Psi \Delta_k^h W_{\tilde{\Phi}})_{L_+^2}| dt' \lesssim d_k^2 2^{-(d-1)k} \|e^\Psi W_{\tilde{\Phi}}\|_{\tilde{L}_{t,\Theta}^2(\mathcal{B}^{\frac{d}{2},0})}^2.$$

Whereas applying Lemma 2.1 yields

$$\begin{aligned} \|e^\Psi \Delta_k^h [T_W^h \nabla_h w^2]_{\tilde{\Phi}}(t)\|_{L_+^2} &\lesssim 2^k \sum_{|k'-k| \leq 4} \|S_{k'-1}^h W_{\tilde{\Phi}}(t)\|_{L_+^\infty} \|e^\Psi \Delta_{k'}^h w_{\tilde{\Phi}}^2\|_{L_t^\infty(L_+^2)} \\ &\lesssim 2^{-\left(\frac{d-1}{2}\right)k} d_k \langle t \rangle^{\frac{1}{4}} \|e^\Psi \partial_y W_{\tilde{\Phi}}(t)\|_{\mathcal{B}^{\frac{d-1}{2},0}} \|e^\Psi w_{\tilde{\Phi}}^2\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{d+1}{2},0})}, \end{aligned}$$

from which, we infer

$$\begin{aligned} (4.11) \quad \int_0^t |(e^\Psi \Delta_k^h [T_W^h \nabla_h w^2]_{\tilde{\Phi}} | e^\Psi \Delta_k^h W_{\tilde{\Phi}})_{L_+^2}| dt' &\lesssim d_k^2 2^{-(d-1)k} (\langle t \rangle^{\frac{3}{2}} - 1)^{\frac{1}{2}} \\ &\quad \times \|e^\Psi \partial_y W_{\tilde{\Phi}}\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{d-1}{2},0})} \|e^\Psi W_{\tilde{\Phi}}\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{d-1}{2},0})} \|e^\Psi w_{\tilde{\Phi}}^2(t)\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{d+1}{2},0})}. \end{aligned}$$

Therefore, in view (4.5), we obtain

$$\begin{aligned} (4.12) \quad \int_0^t |(e^\Psi \Delta_k^h (W \cdot \nabla_h w^2)_{\tilde{\Phi}} | e^\Psi \Delta_k^h W_{\tilde{\Phi}})_{L_+^2}| dt' &\lesssim d_k^2 2^{-(d-1)k} \left(\|e^\Psi W_{\tilde{\Phi}}\|_{\tilde{L}_{t,\Theta}^2(\mathcal{B}^{\frac{d}{2},0})}^2 \right. \\ &\quad \left. + t^{\frac{1}{2}} \|e^\Psi w_{\tilde{\Phi}}^2\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{d-1}{2},0})} \|e^\Psi \partial_y W_{\tilde{\Phi}}\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{d-1}{2},0})} \|e^\Psi W_{\tilde{\Phi}}\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{d-1}{2},0})} \right). \end{aligned}$$

- Estimate of $\int_0^t (e^\Psi \Delta_k^h (\int_0^y \operatorname{div}_h w^1 dy' \partial_y W)_{\tilde{\Phi}} | e^\Psi \Delta_k^h W_{\tilde{\Phi}})_{L_+^2} dt'$

By applying Bony's decomposition and a similar trick of (3.6), we write

$$\begin{aligned} &\|e^\Psi \Delta_k^h (\int_0^y \operatorname{div}_h w^1 dy' \partial_y W)_{\tilde{\Phi}}\|_{L_t^1(L_+^2)} \\ &\lesssim \sum_{k' \geq k-N_0} \int_0^t \left(\|S_{k'-1}^h \operatorname{div}_h w_{\tilde{\Phi}}^1(t')\|_{L_v^1(L_h^\infty)} \|e^\Psi \Delta_{k'}^h \partial_y W_{\tilde{\Phi}}(t')\|_{L_+^2} \right. \\ &\quad \left. + \|\Delta_{k'}^h \operatorname{div}_h w_{\tilde{\Phi}}^1(t')\|_{L_v^1(L_h^2)} \|e^\Psi S_{k'+2}^h \partial_y W_{\tilde{\Phi}}(t')\|_{L_v^2(L_h^\infty)} \right) dt' \\ &\lesssim \sum_{k' \geq k-N_0} \|\langle t' \rangle^{\frac{1}{4}}\|_{L_t^2} \left(\|e^\Psi S_{k'-1}^h \operatorname{div}_h w_{\tilde{\Phi}}^1\|_{L_t^\infty(L_v^2(L_h^\infty))} \|e^\Psi \Delta_{k'}^h \partial_y W_{\tilde{\Phi}}\|_{L_t^2(L_+^2)} \right. \\ &\quad \left. + 2^k \|e^\Psi \Delta_{k'}^h w_{\tilde{\Phi}}^1\|_{L_t^\infty(L_+^2)} \|e^\Psi S_{k'+2}^h \partial_y W_{\tilde{\Phi}}\|_{L_t^2(L_v^2(L_h^\infty))} \right), \end{aligned}$$

from which and Lemma 2.1, we infer

$$\begin{aligned}
& \|e^\Psi \Delta_k^h (\int_0^y \operatorname{div}_h w^1 dy' \partial_y W)_{\tilde{\Phi}}\|_{L_t^1(L_+^2)} \\
& \lesssim (\langle t \rangle^{\frac{3}{2}} - 1)^{\frac{1}{2}} \sum_{k' \geq k - N_0} d_{k'} 2^{-\left(\frac{d-1}{2}\right)k'} \|w_{\tilde{\Phi}}^1\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{d+1}{2},0})} \|\partial_y W_{\tilde{\Phi}}\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{d-1}{2},0})} \\
& \lesssim t^{\frac{1}{2}} d_k 2^{-\left(\frac{d-1}{2}\right)k} \|e^\Psi w_{\tilde{\Phi}}^1\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{d+1}{2},0})} \|e^\Psi \partial_y W_{\tilde{\Phi}}\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{d-1}{2},0})}.
\end{aligned}$$

So that by virtue of (4.5), we obtain

$$\begin{aligned}
(4.13) \quad & \int_0^t |(e^\Psi \Delta_k^h (\int_0^y \operatorname{div}_h w^1 dy' \partial_y W)_{\tilde{\Phi}} | e^\Psi \Delta_k^h W_{\tilde{\Phi}})_{L_+^2}| dt' \\
& \lesssim t^{\frac{1}{2}} d_k^2 2^{-(d-1)k} \|e^\Psi w_{\tilde{\Phi}}^1\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{d+1}{2},0})} \|e^\Psi \partial_y W_{\tilde{\Phi}}\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{d-1}{2},0})} \|e^\Psi W_{\tilde{\Phi}}\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{d-1}{2},0})}.
\end{aligned}$$

- Estimate of $\int_0^t (e^\Psi \Delta_k^h (\int_0^y \operatorname{div}_h W dy' \partial_y w^2)_{\tilde{\Phi}} | e^\Psi \Delta_k^h W_{\tilde{\Phi}})_{L_+^2} dt'$

The estimate of this term is the almost same as that of (4.10). Indeed applying Bony's decomposition (2.3) gives

$$\begin{aligned}
& \int_0^t |(e^\Psi \Delta_k^h (\int_0^y \operatorname{div}_h W dy' \partial_y w^2)_{\tilde{\Phi}} | e^\Psi \Delta_k^h W_{\tilde{\Phi}})_{L_+^2}| dt' \\
& \lesssim \sum_{k' \geq k - N_0} \int_0^t \left(\|S_{k'-1}^h \operatorname{div}_h W_{\tilde{\Phi}}(t')\|_{L_v^1(L_h^\infty)} \|e^\Psi \Delta_{k'}^h \partial_y w_{\tilde{\Phi}}^2(t')\|_{L_+^2} \right. \\
& \quad \left. + \|\Delta_{k'}^h \operatorname{div}_h W_{\tilde{\Phi}}(t')\|_{L_v^1(L_h^2)} \|e^\Psi S_{k'+2}^h \partial_y w_{\tilde{\Phi}}^2(t')\|_{L_v^2(L_h^\infty)} \right) \|e^\Psi \Delta_k^h W_{\tilde{\Phi}}(t')\|_{L_+^2} dt' \\
& \lesssim \sum_{k' \geq k - N_0} 2^{-\left(\frac{d-1}{2}\right)k'} \int_0^t \langle t' \rangle^{\frac{1}{4}} \|e^\Psi \partial_y w_{\tilde{\Phi}}^2(t')\|_{\mathcal{B}^{\frac{d-1}{2},0}} (\|e^\Psi S_{k'-1}^h \operatorname{div}_h W_{\tilde{\Phi}}(t')\|_{L_v^2(L_h^\infty)} \\
& \quad + 2^{k'} \|e^\Psi \Delta_{k'}^h W_{\tilde{\Phi}}(t')\|_{L_+^2}) \|e^\Psi \Delta_k^h W_{\tilde{\Phi}}(t')\|_{L_+^2} dt',
\end{aligned}$$

from which and a similar derivation of (4.10), we arrive at

$$(4.14) \quad \int_0^t (e^\Psi \Delta_k^h (\int_0^y \operatorname{div}_h W dy' \partial_y w^2)_{\tilde{\Phi}} | e^\Psi \Delta_k^h W_{\tilde{\Phi}})_{L_+^2} dt' \lesssim d_k^2 2^{-(d-1)k} \|e^\Psi W_{\tilde{\Phi}}\|_{\tilde{L}_{t,\Theta}^2(\mathcal{B}^{\frac{d}{2},0})}^2.$$

Finally in view of (4.2) and (4.3), one has

$$\begin{aligned}
& \int_0^t |(e^\Psi \Delta_k^h (\int_0^y \operatorname{div}_h W dy')_{\tilde{\Phi}} \partial_y u^s | e^\Psi \Delta_k^h W_{\tilde{\Phi}})_{L_+^2}| dt' \\
& \lesssim 2^k \int_0^t \langle t' \rangle^{\frac{1}{4}} \|e^\Psi \partial_y u^s(t')\|_{L_v^2} \|e^\Psi \Delta_k^h W_{\tilde{\Phi}}(t')\|_{L_+^2}^2 dt',
\end{aligned}$$

from which and Definitions 2.2 and 2.3, we infer

$$(4.15) \quad \int_0^t |(e^\Psi \Delta_k^h (\int_0^y \operatorname{div}_h W dy')_{\tilde{\Phi}} \partial_y u^s | e^\Psi \Delta_k^h W_{\tilde{\Phi}})_{L_+^2}| dt' \lesssim d_k^2 2^{-(d-1)k} \|e^\Psi W_{\tilde{\Phi}}\|_{\tilde{L}_{t,\Theta}^2(\mathcal{B}^{\frac{d}{2},0})}^2.$$

Now let us complete the proof of Theorem 1.1.

Proof of the uniqueness part of Theorem 1.1. Resuming the Estimates (4.10) and (4.12)–(4.15) into (4.6), then taking square root of the resulting inequality and multiplying it by $2^{\left(\frac{d-1}{2}\right)k}$, we thus obtain, by summing over the final inequality for $k \in \mathbb{Z}$, that

$$\begin{aligned} & \|e^\Psi W_{\tilde{\Phi}}\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{d-1}{2},0})} + c\sqrt{\lambda}\|e^\Psi W_{\tilde{\Phi}}\|_{\tilde{L}_{t,\tilde{\Theta}}^2(\mathcal{B}^{\frac{d}{2},0})} + \|e^\Psi \partial_y W_{\tilde{\Phi}}\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{d-1}{2},0})} \\ & \leq C\|e^\Psi W_{\tilde{\Phi}}\|_{\tilde{L}_{t,\tilde{\Theta}}^2(\mathcal{B}^{\frac{d}{2},0})} + \frac{1}{2}\|e^\Psi \partial_y W_{\tilde{\Phi}}\|_{\tilde{L}_t^2(\mathcal{B}^{\frac{d-1}{2},0})} \\ & \quad + Ct^{\frac{1}{2}}\left(\|e^\Psi w_{\Phi^1}^1\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{d-1}{2},0})} + \|e^\Psi w_{\Phi^2}^2\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{d-1}{2},0})}\right)^{\frac{1}{2}}\|e^\Psi W_{\tilde{\Phi}}\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{d-1}{2},0})}. \end{aligned}$$

Hence, by taking λ large enough and t being sufficiently small in the above inequality, we deduce that $\|e^\Psi W_{\tilde{\Phi}}\|_{\tilde{L}_t^\infty(\mathcal{B}^{\frac{d-1}{2},0})} = 0$ for some small time t . The uniqueness for whole time of existence can be deduced by a continuous argument. This completes the proof of Theorem 1.1. \square

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(P. ZHANG) ACADEMY OF MATHEMATICS & SYSTEMS SCIENCE AND HUA LOO-KENG KEY LABORATORY OF MATHEMATICS, THE CHINESE ACADEMY OF SCIENCES, BEIJING 100190, CHINA

E-mail address: zp@amss.ac.cn

(Z. ZHANG) SCHOOL OF MATHEMATICAL SCIENCE, PEKING UNIVERSITY, BEIJING 100871, P. R. CHINA

E-mail address: zfzhang@math.pku.edu.cn